Asymptotically Optimal Locally Private Heavy Hitters

via Parameterized Sketches

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Frequency Estimation

- A set $\mathcal{U}$ of n users and a server.
- Each user $u \in [n]$ holds an element $v^{(u)}$ from some data domain $\mathcal{D}$ of size d.
- Each user reports their element.
- Two frequency estimation tasks for server:

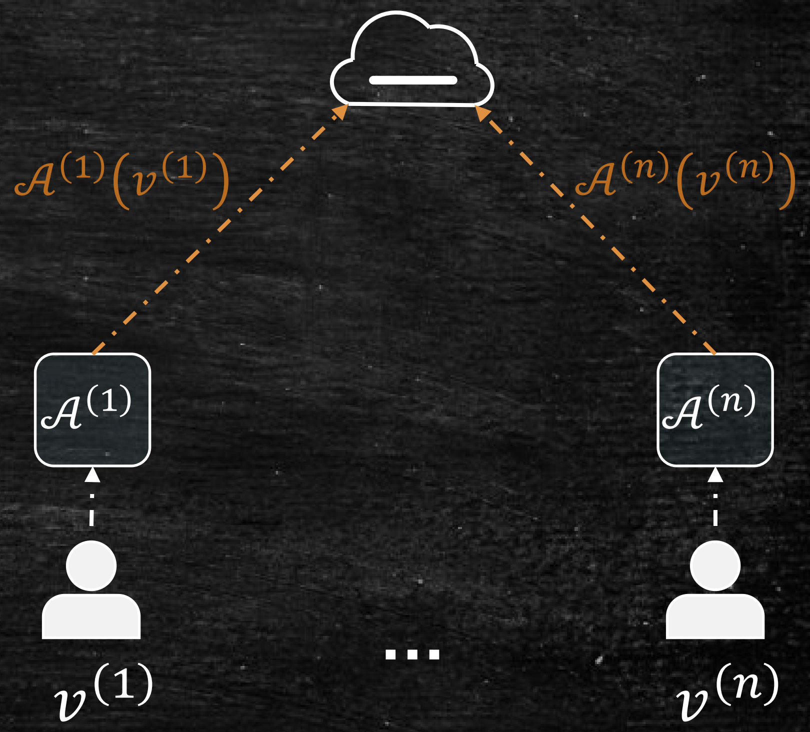
1. Frequency Oracle $\mathcal{A}_{oracle}$:
   - (Informally) given $v \in \mathcal{D}$, return an estimate of v's frequency in $\mathcal{U}$.
2. Succinct Histogram $\mathcal{A}_{hist}$:
   - (Informally) return a set of elements with high frequencies amongst $\mathcal{U}$, known as *heavy hitters*, together with their frequency estimates.

Local Differential Privacy

- To protect sensitive information, the users don't report their elements directly.
- Each user u perturbs $v^{(u)}$ with a local randomizer $\mathcal{A}^{(u)}: \mathcal{D} \to \mathcal{Y}$ before reporting.
- Server preforms the frequency estimation tasks based on the noisy reports.
- $\mathcal{A}^{(u)}$ is ε-local differentially private: for all $v, v' \in \mathcal{D}$ and all (measurable) $Y \subseteq \mathcal{Y}$,

$$\Pr[\mathcal{A}^{(u)}(v) \in Y] \leq e^{\epsilon} \cdot \Pr[\mathcal{A}^{(u)}(v') \in Y]$$

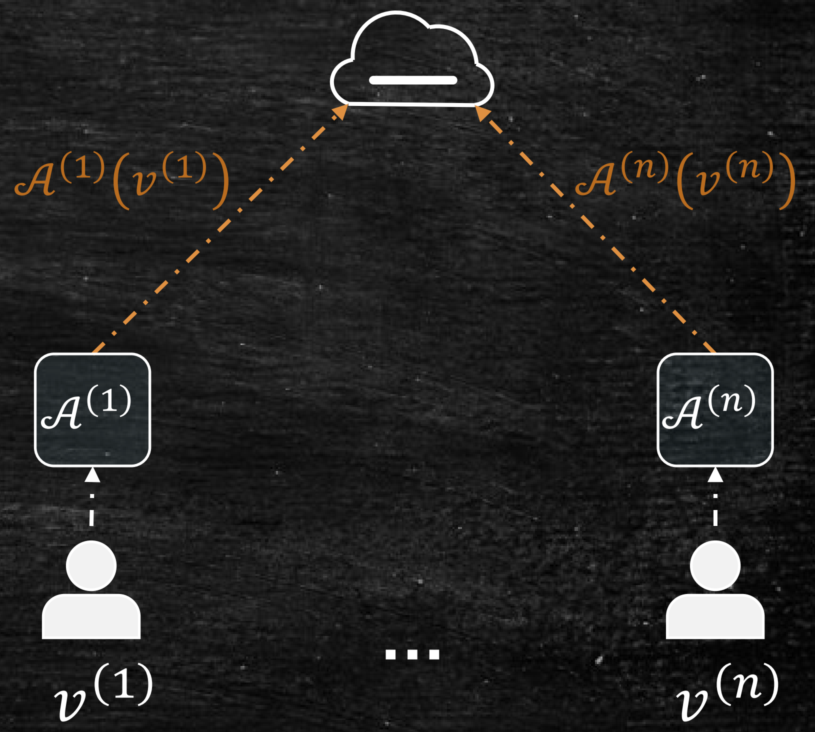
  - The output distribution of $\mathcal{A}^{(u)}$ varies little with the input.



Challenge

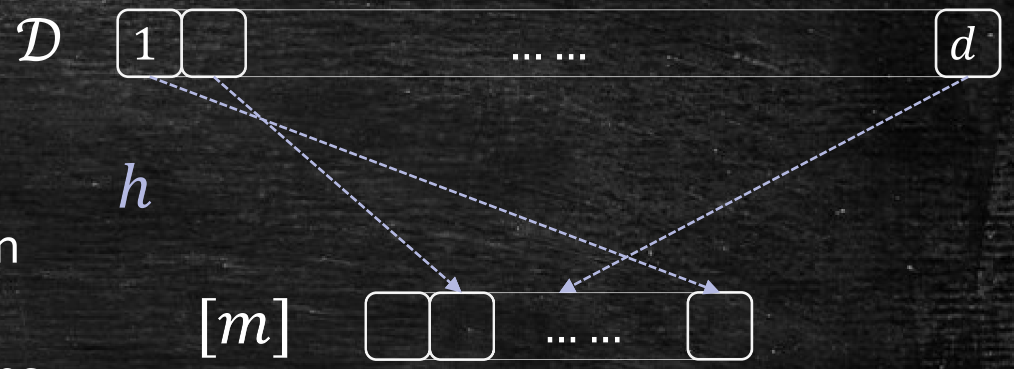
- The server-side algorithm's running time/memory usage may scale linearly with data domain size d.

- Example of concern: popular URLs with length up to 20 characters *(Fanti et al., 2016)*
  - URL characters can include digits (0-9), letters(A-Z, a-z), and a few special characters ("-" , "." , " " , "~ ").
  - Hence $d = 66^{20} \geq 10^{36}$

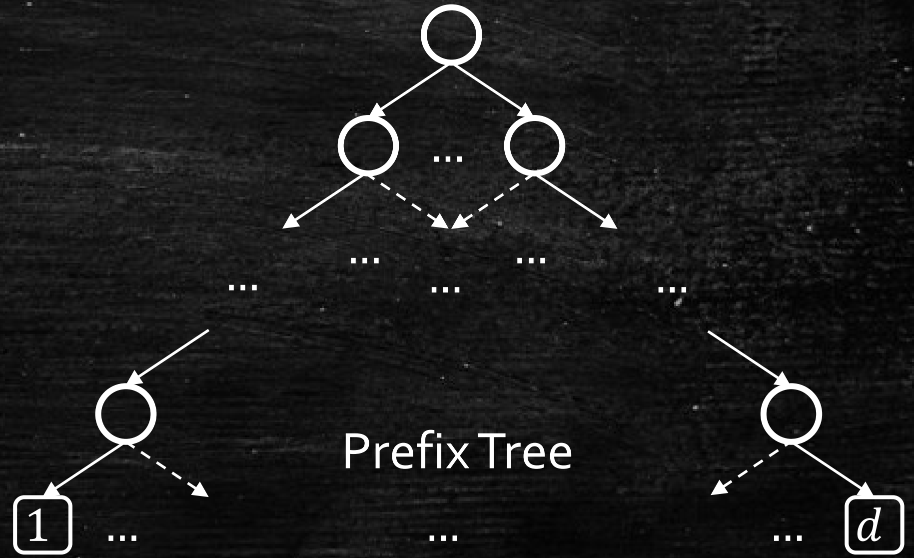


Existing Deployed Solutions

- *Sketching.*
  - Reduce the size of $\mathcal{D}$ for $\mathcal{A}_{oracle}$.
  - Pick a hash function h, which maps elements in $\mathcal{D}$ to a smaller domain $[m]$, for some $m \in \mathbb{N}^+$.
  - Repetitions may be required to handle collisions.



- *Hierarchical Searching.*
  - Avoid inspecting each element in $\mathcal{D}$.
  - Encode elements in $\mathcal{D}$ as strings, over which a prefix tree is constructed.
  - If an element in $\mathcal{D}$ is heavy, so are its prefixes.
  - Heavy hitters are identified top down.



Existing Work Comparison

For failure probability β,

- *Sketching* $\mathcal{A}_{oracle}$:
  - *FreqOracle* (Bassily et al., 17)
  - Estimation error:

    $$O\left((1/\epsilon) \cdot \sqrt{n \cdot \ln(n/\beta)}\right)$$

  - Server running time: $\tilde{O}(n)$
  - Server memory: $\tilde{O}(\sqrt{n})$
- *Hierarchical Searching* $\mathcal{A}_{hist}$:
  - *TreeHist* (Bassily et al., 17)
  - Estimation error:

    $$O\left((1/\epsilon) \cdot \sqrt{n \cdot \ln(d) \cdot \ln(n/\beta)}\right)$$

  - Server running time: $\tilde{O}(n)$
  - Server memory: $\tilde{O}(\sqrt{n})$

- *Easy to implement; Low time complexity and memory usage.*
  - *Their variants are studied experimentally* (Cormode et al. 21).
  - *Algorithms following this vein perform well in practice.*
- *Sup-optimal errors.*

Existing Work Comparison

- *Theoretical optimal error.*
- *Running time and memory usage depend on d.*

- *Non-Sketching $\mathcal{A}_{oracle}$:*
  - *HRR (Nguyên et al., 16; Cormode et al., 19)*
  - Estimation error (matches lower bound):

    $$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(1/\beta)}\right)$$

  - Server running time: $O(n+d)$
  - Server memory: $O(d)$

- *Theoretical optimal error.*
- *Due to the sophistication of error-correcting codes, it has not been implemented.*

- *Error-Correcting Code $\mathcal{A}_{hist}$:*
  - *PrivateExpanderSketch (Bun et al., 19)*
  - Estimation error (matches lower bound):

    $$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(d/\beta)}\right)$$

  - Server running time: $\tilde{O}(n)$
  - Server memory: $\tilde{O}(\sqrt{n})$

Can We Close the Gaps?

1. Are theoretical error guarantees of existing approaches *sketching* $\mathcal{A}_{oracle}$ and *hierarchical searching* $\mathcal{A}_{hist}$ best possible?

2. Or can we obtain algorithms of this type that achieve optimal error guarantee?

New $\mathcal{A}_{oracle}$ **HadaOracle** & $\mathcal{A}_{hist}$ **HadaHeavy**

- *Sketching $\mathcal{A}_{oracle}$*:
  - *FreqOracle* *(Bassily et al., 17)*
  - Estimation error:
    $$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(n/\beta)}\right)$$
  - Server running time: $\tilde{O}(n)$
  - Server memory: $\tilde{O}(\sqrt{n})$

- *Non-Sketching $\mathcal{A}_{oracle}$*:
  - *HRR* *(Nguyễn et al., 16; Cormode et al., 19)*
  - Estimation error (matches lower bound):
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  - Server running time: $O(n+d)$
  - Server memory: $O(d)$

- *Hierarchical Searching $\mathcal{A}_{hist}$*:
  - *TreeHist* *(Bassily et al., 17)*
  - Estimation error:
    $$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(d)\cdot\ln(n/\beta)}\right)$$
  - Server running time: $\tilde{O}(n)$
  - Server memory: $\tilde{O}(\sqrt{n})$

if $\beta = n^{-c}$

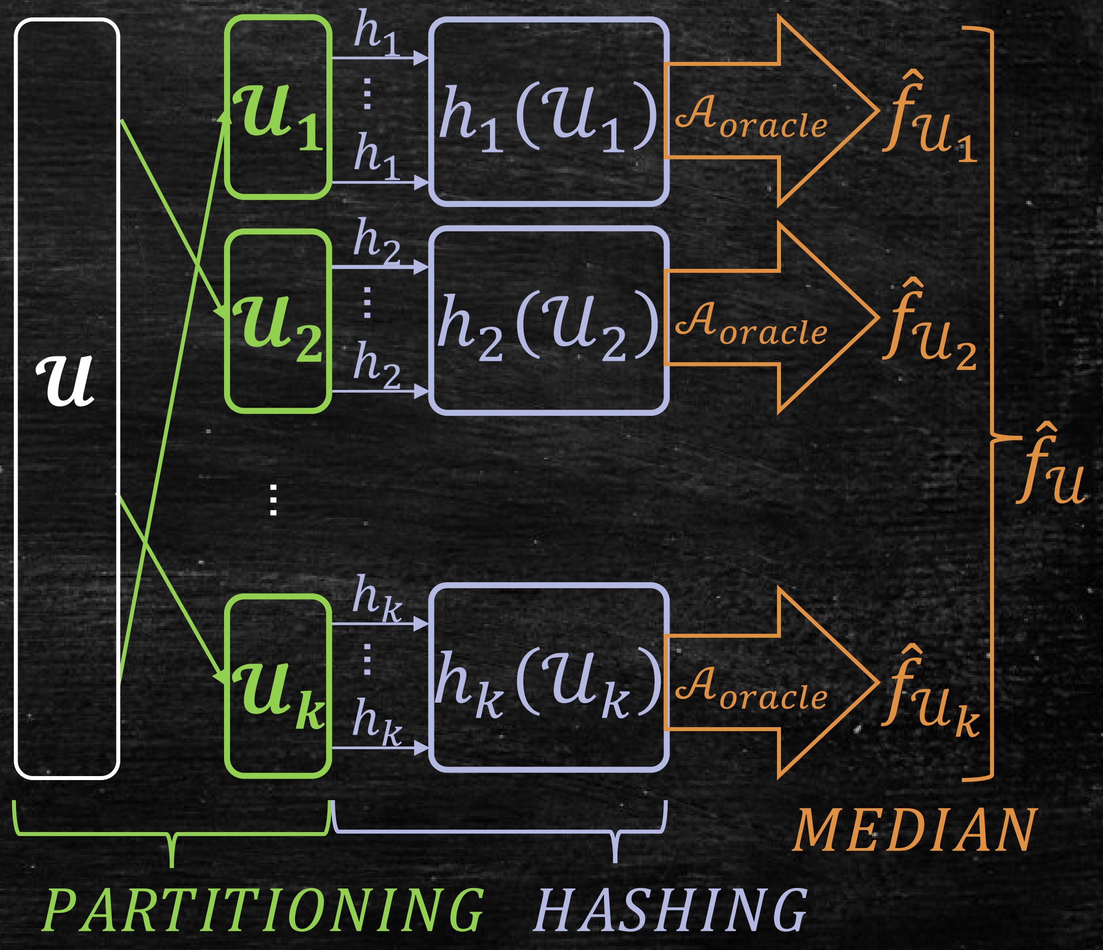
- *Error-Correcting Code $\mathcal{A}_{hist}$*:
  - *PrivateExpanderSketch* *(Bun et al., 19)*
  - Estimation error (matches lower bound):
    $$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(d/\beta)}\right)$$
  - Server running time: $\tilde{O}(n)$
  - Server memory: $\tilde{O}(\sqrt{n})$

Summary of Results

	Performance Metric	Server Time	Server Mem	Worst-Case Error	Lower Bound
FO	HadaOracle (*this work*)	$\tilde{O}(n)$	$\tilde{O}(\sqrt{n})$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(1/\beta)}\right)$	$\Omega\left(\frac{1}{\epsilon}\sqrt{n\ln\frac{1}{\beta}}\right)$
	HRR (*Nguyên et al., 16; Cormode et al., 19*)	$\tilde{O}(d)$	$\tilde{O}(d)$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(1/\beta)}\right)$	
	FreqOracle (*Bassily et al., 17*)	$\tilde{O}(n)$	$\tilde{O}(\sqrt{n})$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(n/\beta)}\right)$	
	Hashtogram (*Bassily et al., 17*)	$\tilde{O}(n)$	$\tilde{O}(\sqrt{n})$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(n/\beta)}\right)$	
S-Hist	HadaHeavy (*this work*)	$\tilde{O}(n)$	$\tilde{O}(\sqrt{n})$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(d)\cdot\left(1+\frac{\ln(1/\beta)}{\ln n}\right)}\right)$	$\Omega\left(\frac{1}{\epsilon}\sqrt{n\ln\frac{d}{\beta}}\right)$
	TreeHist (*Bassily et al., 17*)	$\tilde{O}(n)$	$\tilde{O}(\sqrt{n})$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot(\ln d)\cdot\ln(n/\beta)}\right)$	
	PrivateExpanderSketch (*Bun et al., 19*)	$\tilde{O}(n)$	$\tilde{O}(\sqrt{n})$	$O\left((1/\epsilon)\cdot\sqrt{n\cdot\ln(d/\beta)}\right)$	

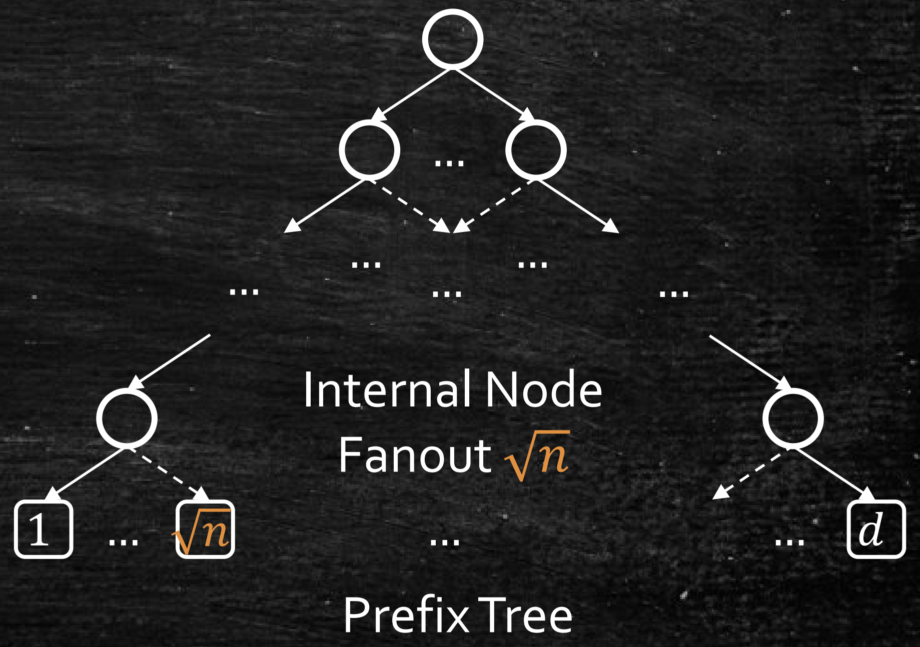
HadaOracle

- Randomly partition users $\mathcal{U}$ into subsets $\mathcal{U}_1, \ldots, \mathcal{U}_k$, where $k \in \Theta(\ln(1/\beta))$, analysis of which is based on *martingale concentration inequalities*.
- For each $i \in [k]$:
  - pick a pairwise independent hash function $h_i : \mathcal{D} \to [m]$, where $m \in \Theta(\sqrt{n})$.
  - for each $u \in \mathcal{U}_i$, replace their element $v^{(u)}$ by $h_i(v^{(u)})$.
  - denote the hashed elements of users in $\mathcal{U}_i$ by $h_i(\mathcal{U}_i)$.
- Construct $\mathcal{A}_{oracle}$(*HRR*) over all $h_i(\mathcal{U}_i)$:
  - for a query $v \in D$, return the scaled median of its frequency estimates.



HadaHeavy

- Encode elements in $\mathcal{D}$ as strings with alphabet size $\sqrt{n}$.
- Construct the prefix tree.
  - It has depth $L \doteq 2 \cdot \frac{\ln d}{\ln n}$.
  - Randomly split the users $\mathcal{U}$ into L groups, $U_1, U_2, \ldots, U_L$.
  - Build a *HadaOracle* for nodes with depth i, based on users U_i, for each $i \in [L]$.
- Heavy hitters are identified top down.
  - If a node is heavy, inspect its children to find possible heavy nodes.



THANK YOU

REFERENCES

- *G. C. Fanti, et.al, "Building a RAPPOR with the unknown: Privacy-preserving learning of associations and data dictionaries," Priv. Enhancing Technol., 2016.*
- *G. Cormode, et.al, "Answering range queries under local differential privacy," VLDB Endow., 2019.*
- *G. Cormode, et.al, "Frequency estimation under local differential privacy [experiments, analysis and benchmarks]," CoRR, 2021.*
- *M. Bun, et.al, "Heavy hitters and the structure of local privacy," ACM Trans. Algorithms, 2019.*
- *R. Bassily, et.al, "Practical locally private heavy hitters," NeurIPS., 2017,*
- *T. T. Nguyên, et.al, "Collecting and analyzing data from smart device users with local differential privacy," CoRR., 2016.*