On Convergence of Lookahead in Smooth Games

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Smooth Game Optimization

- A smooth game between n players is defined as a set of smooth functions $\{f_i\}_{i=1}^{n}$ where each function f_i: $\mathbb{R}^d \rightarrow \mathbb{R}$ represents player i's utility given a strategy profile $x = (x_i, x_{-i}) \in \mathbb{R}^d$ of the game.

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- A variety of modern ML problems can be formulated as a game:
  - Generative Adversarial Networks
  - Adversarial Training
  - Robust Optimization
  - Multi-Agent Reinforcement Learning

The Key Challenge: Non-convergence

- The holy grail of game optimization: finding a Nash equilibrium
  - A strategy profile where no player has unilateral incentive to change
  - $x^* \in \mathbb{R}^d$ such that $f_i(x^*) \leq f(x_i, x^*_{-i}), \forall x_i \in \mathbb{R}^{d_i}$ for each $i = 1, \dots, n$.

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- However, gradient methods often **fail to converge in games**
- Example: gradient descent on game $f_1(x, y) = xy = -f_2(x, y)$

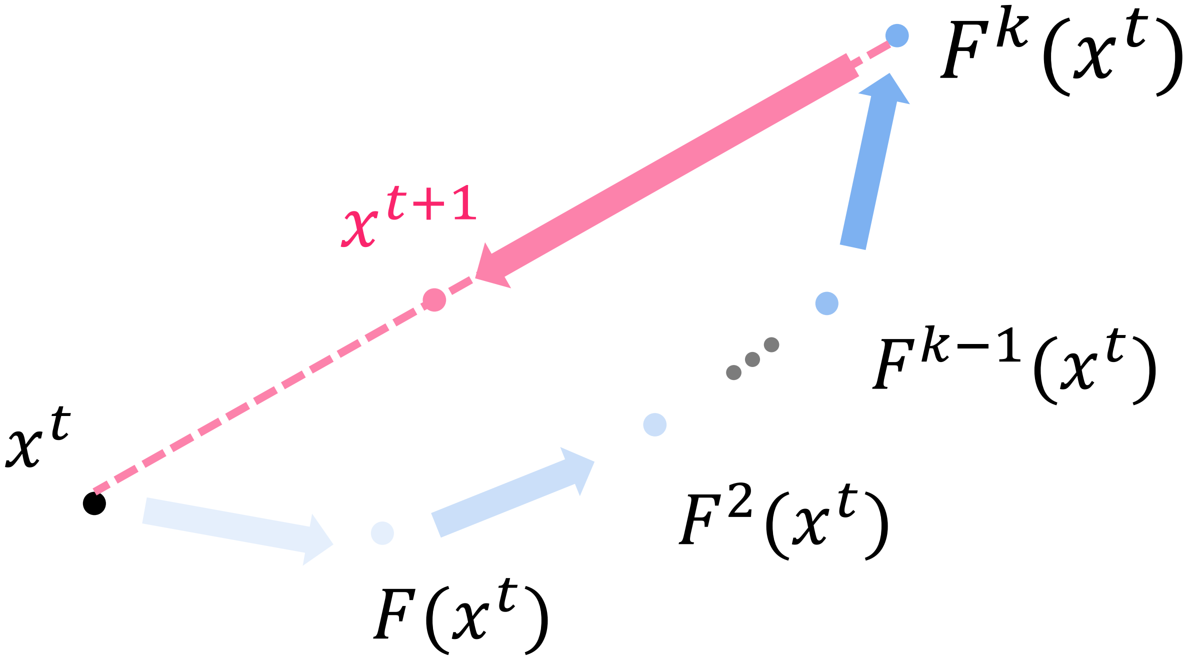
Our Contribution

- We show that Lookahead (Zhang et al., 2019) [1] provides a general mechanism for **local stabilization and acceleration in games**
- Our result naturally transfers to global guarantees in bilinear games

[1] Zhang et al. NeurIPS 2019. Lookahead Optimizer: k steps forward, 1 step back.

Lookahead Optimizer

- Lookahead wraps around a base optimizer F and takes a “backward” step for each k “forward” steps.



Lookahead Optimizer

- Specifically, given a base dynamics $F_{\mathcal{A}}: \mathbb{R}^d \to \mathbb{R}^d$ induced by an optimizer $\mathcal{A}$, its Lookahead dynamics $G_{\mathcal{A}}: \mathbb{R}^d \to \mathbb{R}^d$ with a period $k \in \mathbb{N}$ and a rate $\alpha \in (0, 1)$ is defined as:

$$G_{\mathcal{A}}(x^t) \coloneqq x^{t+1} = (1 - \alpha)x^t + \alpha F_{\mathcal{A}}^k(x^t)$$

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- Notably, Chavdarova et al., 2021 [2] report a promising empirical observation that **Lookahead significantly improves GAN training**

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Key Idea

- Local convergence of a dynamics F around a stationary point x^* can be characterized by the largest eigenvalue of the Jacobian $\nabla_x F(x^*)$

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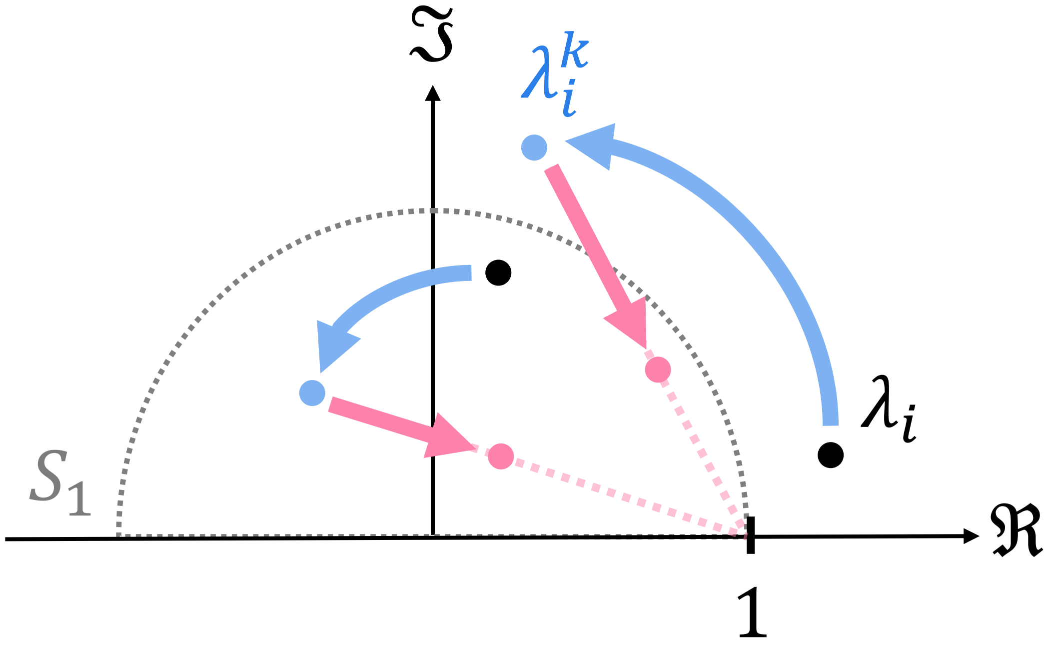
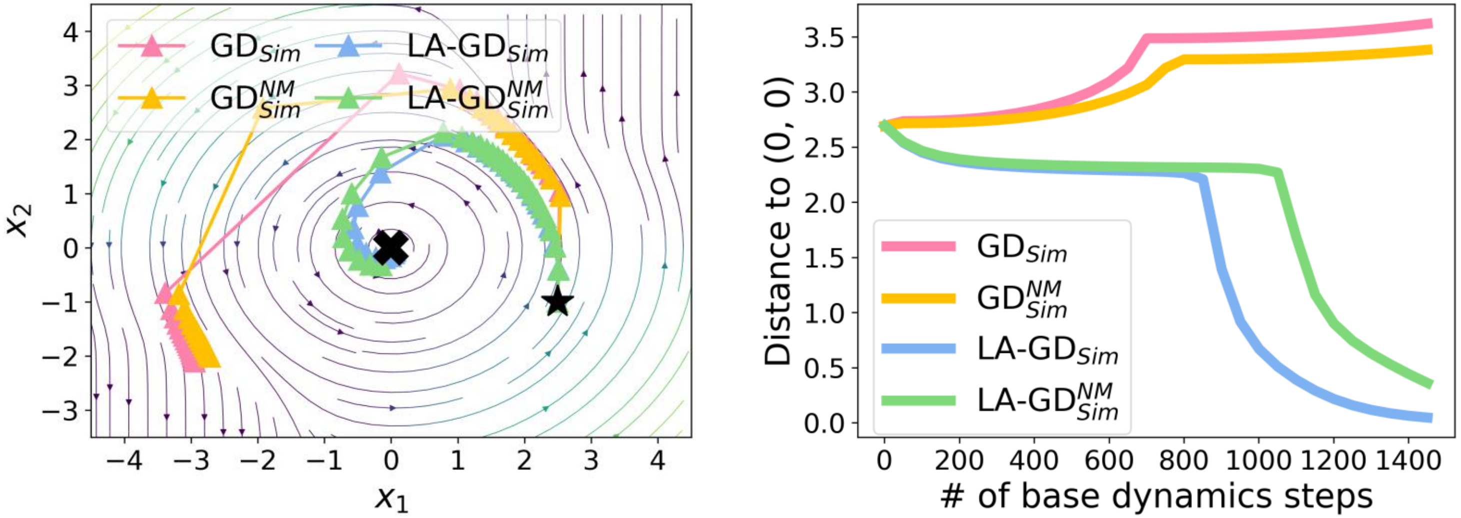
- Local convergence of a dynamics F around a stationary point x^* can be characterized by the largest eigenvalue of the Jacobian $\nabla_x F(x^*)$
- Each Lookahead iteration can be interpreted as **a geometric transformation of the eigenvalues λ_i** of the Jacobian $\nabla_x F(x^*)$



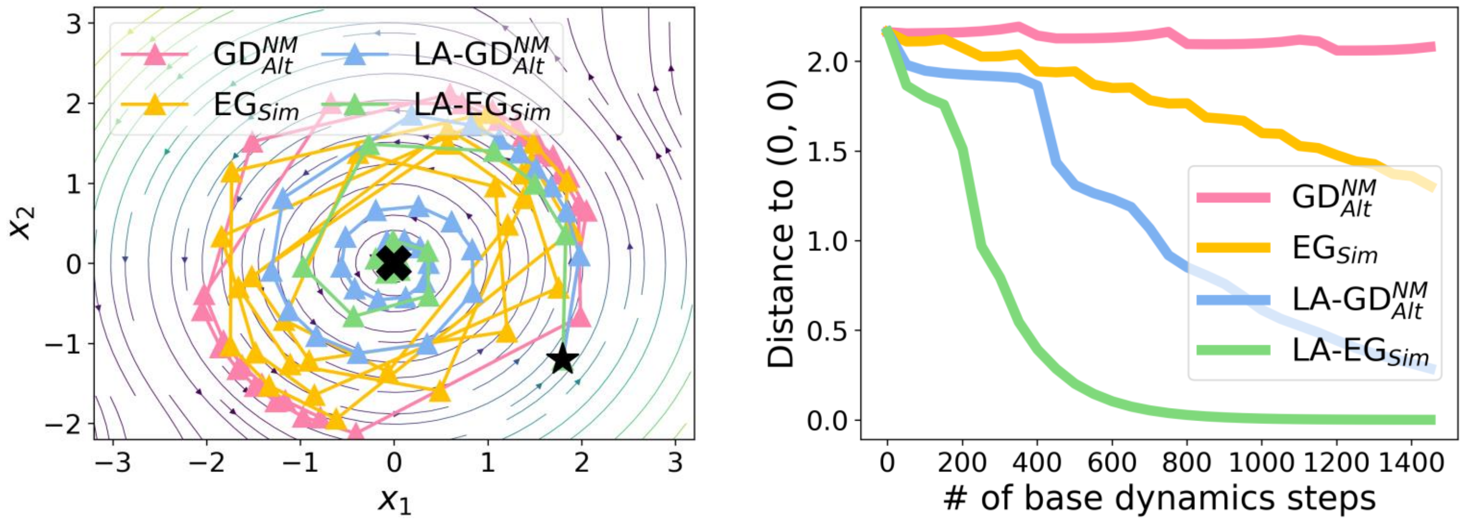
Main Result – Local Stabilization

- **(Theorem 4; informal)** Let $x^* \in \mathbb{R}^n$ be a stationary point of a dynamics F who fails to locally converge to x^*. Then, under certain assumptions on the eigenvalues of $\nabla_x F(x^*)$, its Lookahead dynamics with $k \in \mathbb{N}$ and $\alpha \in (0,1)$ **locally converges to $\boldsymbol{x^*}$ if $\boldsymbol{k \in (\beta_1, \beta_2)}$ and $\boldsymbol{\alpha}$ is small enough,** where $\beta_1, \beta_2 > 0$ are some constants that depend on $\nabla_x F(x^*)$.

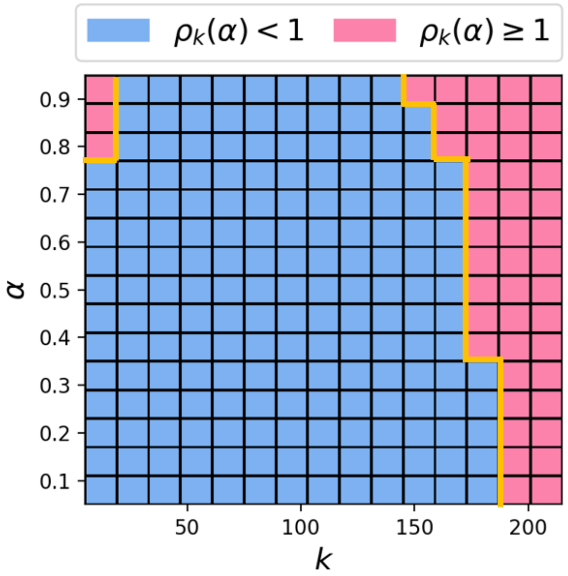


Main Result – Local Acceleration

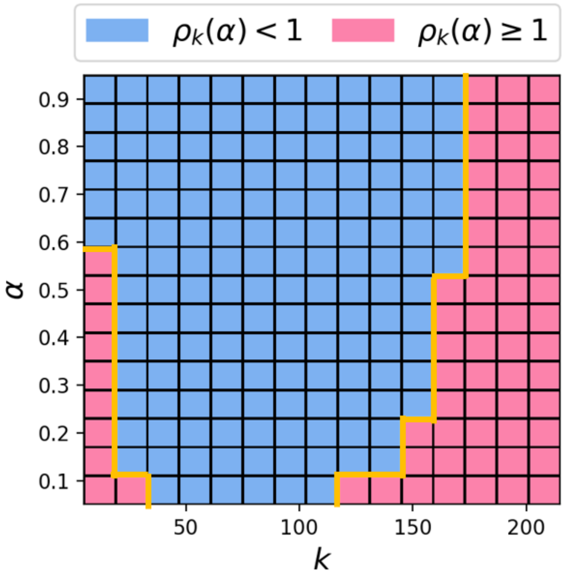
- **(Theorem 5; informal)** Let $x^* \in \mathbb{R}^n$ be a stationary point of a dynamics F who locally converges to x^*. Then, under a mild assumption on the eigenvalues of $\nabla_x F(x^*)$, the local convergence rate of its Lookahead dynamics with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ **improves upon *F* if $\boldsymbol{k} \in (\boldsymbol{\beta_1}, \boldsymbol{\beta_2})$ and $\boldsymbol{\alpha}$ is large enough,** where $\beta_1, \beta_2 > 0$ are some constants that depend on $\nabla_x F(x^*)$.



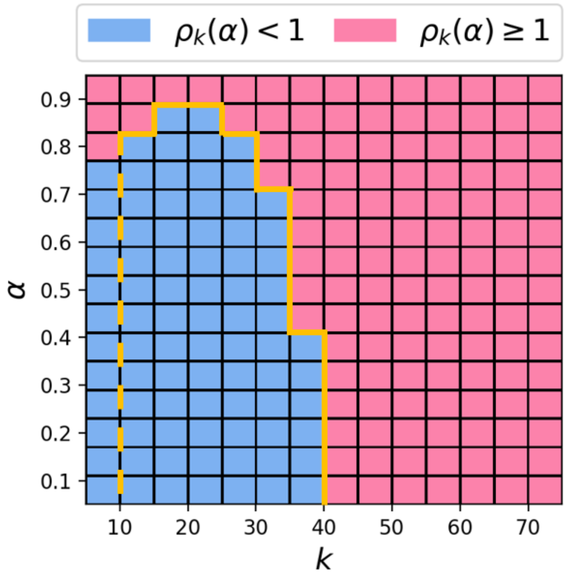
Experiment: Numerical Evaluation



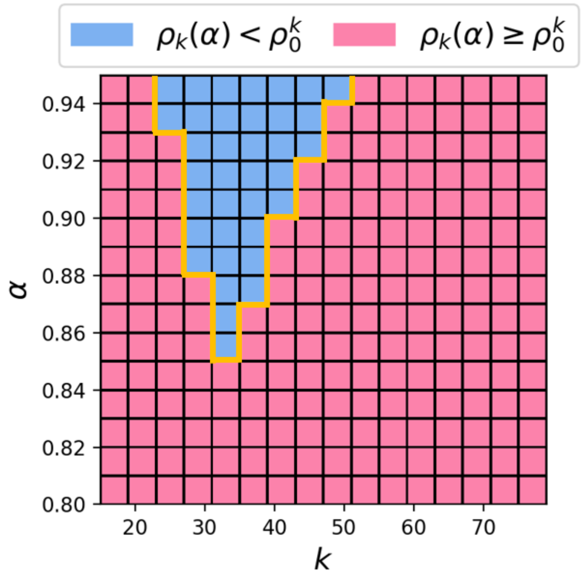
Theorem 4: Stabilization



Theorem 5: Acceleration



Corollary 7: Stabilization

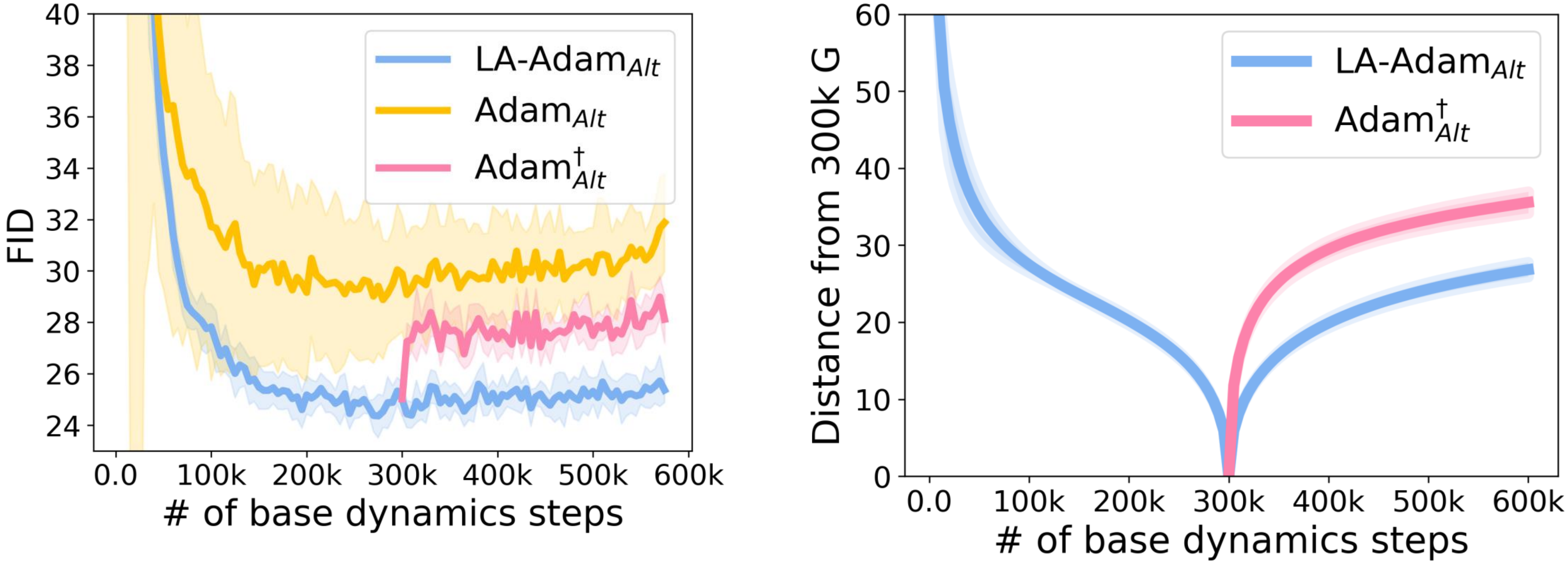


Corollary 8: Acceleration

Nonlinear game (Eq. 9)

Bilinear game (Eq. 10)

Experiment: Stabilization Effect on GANs



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- Intriguing research directions:
  - “better” eigenvalue transformation?
  - Global analysis for stochastic & non-bilinear games

Thank You!

Code available at https://github.com/kuc2477/la-dynamics-in-games