A Statistical Perspective on Coreset Density Estimation

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Density Estimation

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- **Kernel density estimator (KDE):**

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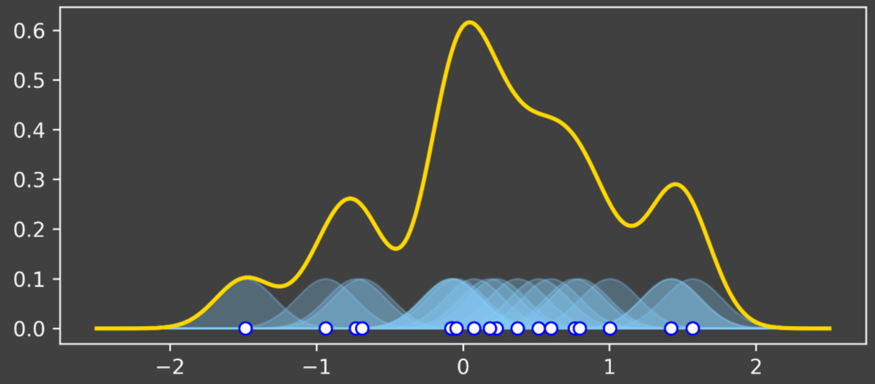

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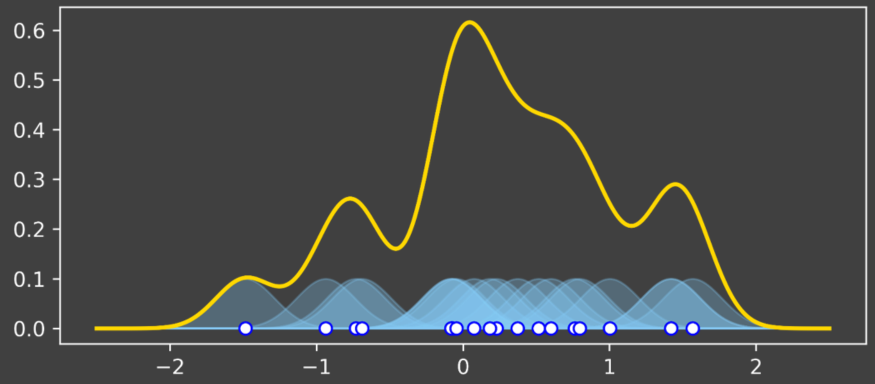

- **Fact:** KDE has rate of estimation $n^{-\frac{\beta}{2\beta+d}}$ over Hölder β densities.
- **Question:** Can we improve on computational aspects?

KDE

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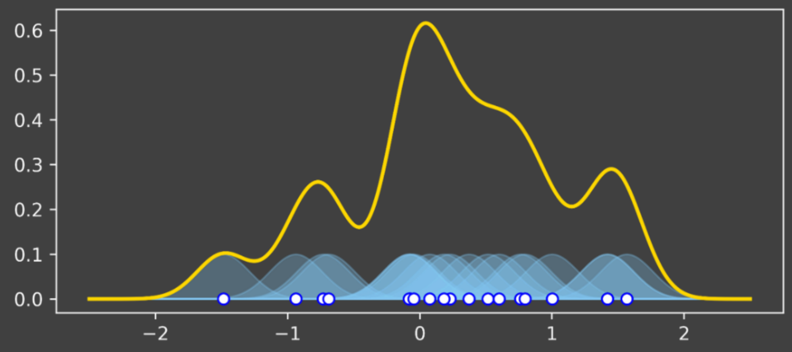

KDE

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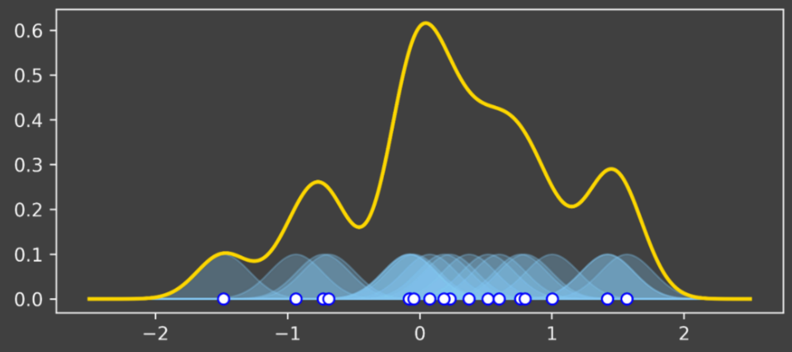
$$= \mathbb{E}_{X\sim\mathbb{P}_n}\left[K_h(X - y)\right]$$

empirical measure



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$$= \mathbb{E}_{X\sim\mathbb{P}_n}[K_h(X - y)] \longrightarrow \mathbb{E}_{X\sim\mathbb{P}_c}[K_h(X - y)]$$

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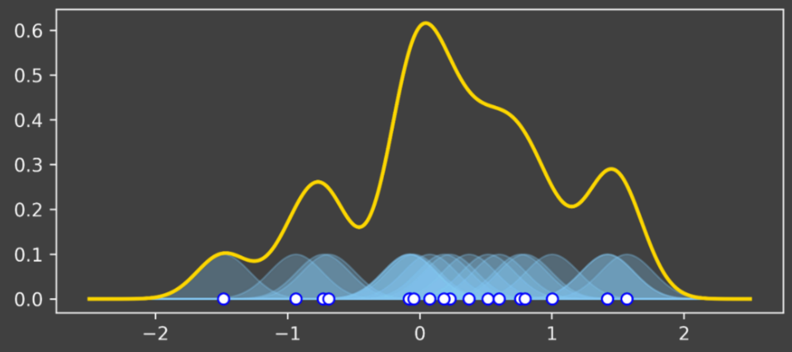

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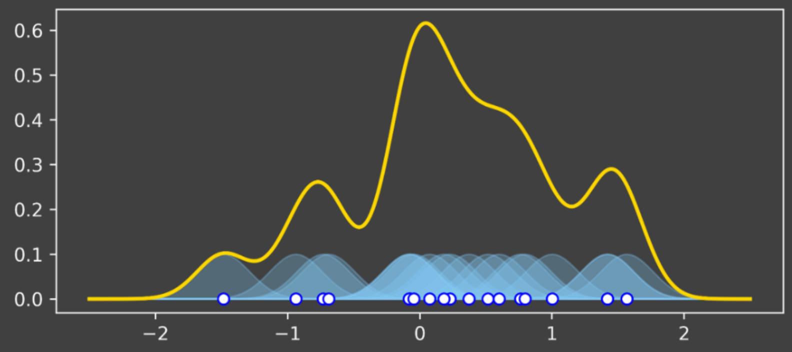

Coreset: a weighted subset of the data

$$\mathbb{E}_{X\sim\mathbb{P}_C}[K_h(X - y)]$$

coreset measure

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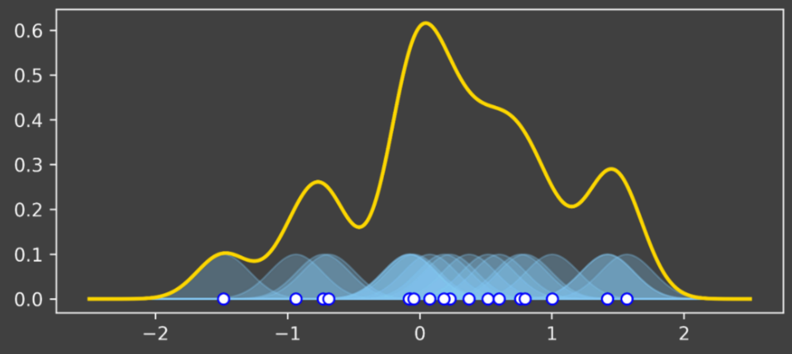


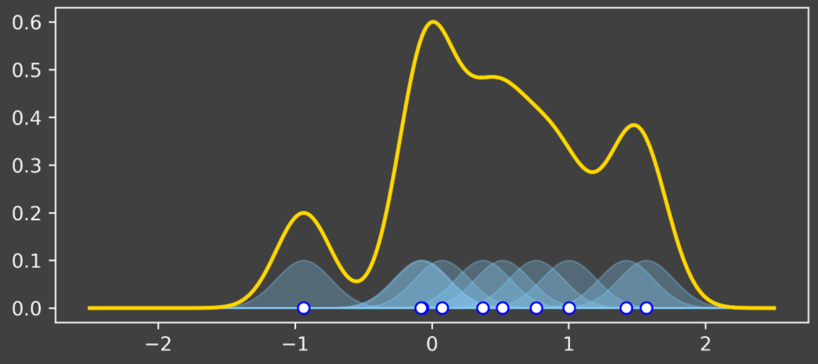
Coreset KDE

Coreset: a weighted subset of the data

$$\hat{f}_C(y) = \sum_{X_j\in C} \lambda_j K_h\left(X_j - y\right)$$

$$= \mathbb{E}_{X\sim\mathbb{P}_C}[K_h(X - y)]$$

coreset measure



What is the rate of estimation of coreset KDEs?

$$\hat{f}_C(y) = \sum_{X_j \in C} \lambda_j \, K_h\left(X_j - y\right)$$

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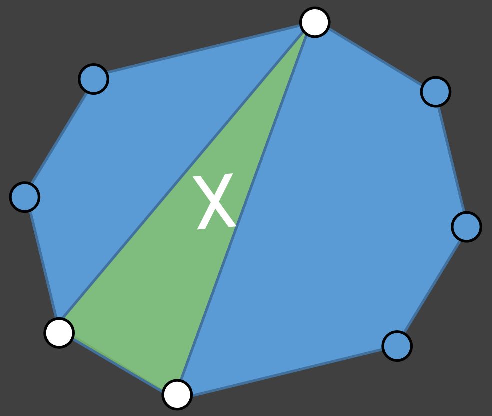
- By Carathéodory's theorem, can take $T = \Omega(|\boldsymbol{C}|)$



Analysis

- By Carathéodory, can find $\{\lambda_j\}$ and $\boldsymbol{C}$ such that

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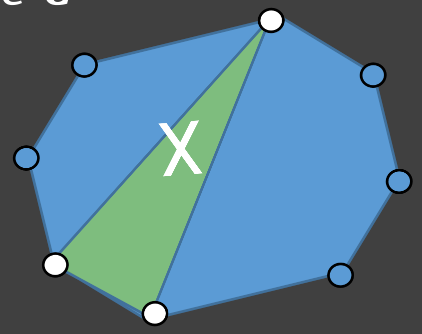

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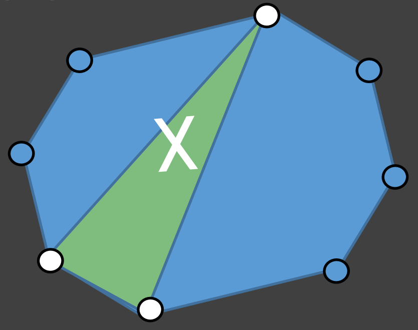

$$\left|\hat{f}(y_0) - \hat{f}_{\boldsymbol{C}}(y_0)\right|$$

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coreset
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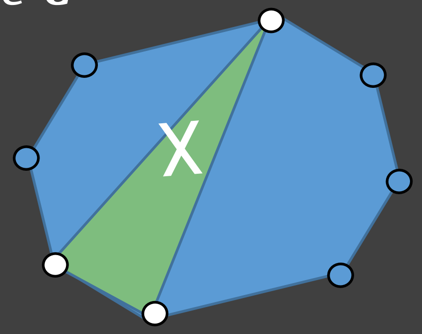

$$\left|\hat{f}(y_0) - \hat{f}_{\boldsymbol{C}}(y_0)\right| \lesssim \left|\sum_{\omega}\left(\frac{1}{n}\sum_{j=1}^{n} e^{i\omega X_j} - \sum_{X_j\in\boldsymbol{C}} \lambda_j e^{i\omega X_j}\right) F[K_h](\omega)\right|$$

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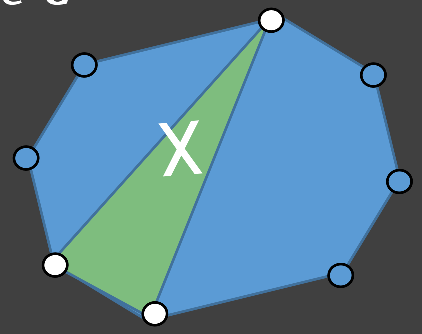

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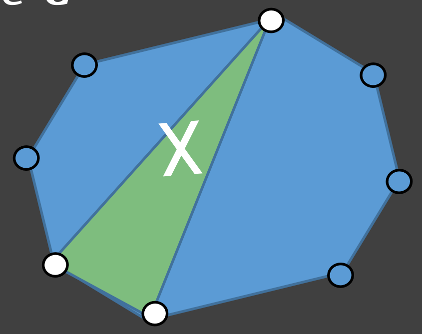

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Small if K is smooth and $T > h^{-1-\varepsilon}$

Main Result

Theorem For appropriate kernel, the Carathéodory KDE achieves the minimax rate $n^{-\frac{\beta}{2\beta+d}}$ with a coreset of size

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We also show *any coreset procedure* requires at least $n^{\frac{d}{2\beta+d}}$ points to achieve minimax rate