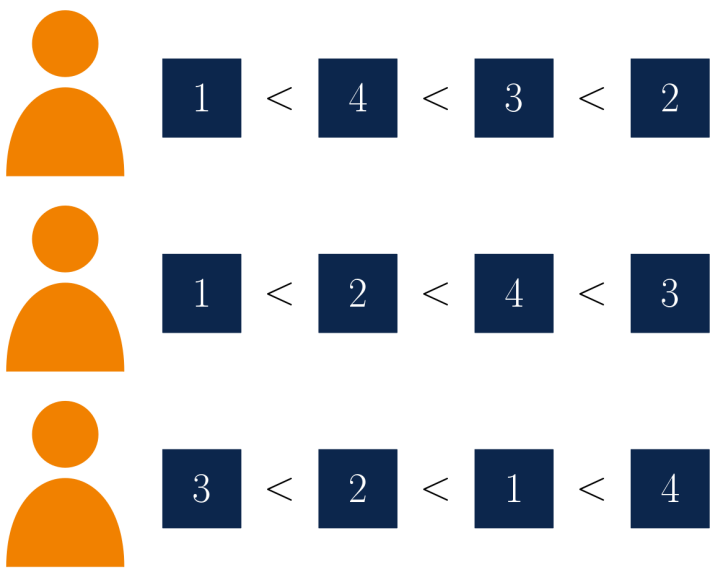
Aggregating Incomplete and Noisy Rankings

Dimitris Fotakis and Alkis Kalavasis and Konstantinos Stavropoulos

National Technical University of Athens

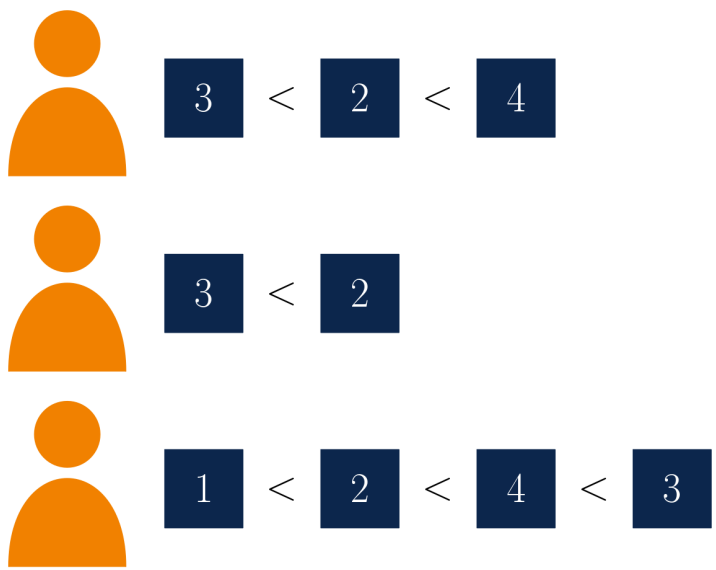
Problem

- Aggregating collections of ranked preferences: *analyze statistical ranking models* (e.g. [Mal57], [Thu27], [Smi50], [BT52], [Pla75], [Luc12]).



Problem

- Incomplete input rankings: *analyze incomplete ranking models.*



Mallows Model

- Extensively studied (e.g. [BM09], [AF98], [CPS13], [LM18], [BFZ19], [FV86], [MM03], [LL03]).
- Central (ground truth) ranking $\pi_0 \leftrightarrow$ ideal output of aggregation.
- Exponential decay to the (Kendall Tau) distance from π_0:

$$\Pr[\pi|\pi_0, \beta] \propto \exp(-\beta d_{KT}(\pi_0, \pi)), \forall \pi \in \mathfrak{S}_n, \tag{1}$$

$\beta > 0$: spread parameter,
$d_{KT}(\pi, \pi') = \#\text{inversions between } \pi, \pi'$.

Selective Mallows Model

- Generates incomplete rankings.
- Selection sets: $\mathcal{S} = (S_1, \ldots, S_r)$, where $S_i \subseteq [n]$.
- Exponential decay to the distance from sub-rankings of π_0:

$$\Pr[(\pi_1, \ldots, \pi_r) | \pi_0, \beta, \mathcal{S}] \propto \prod_{i \in [r]} \exp(-\beta d_{KT}(\pi_0|_{S_i}, \pi_i)), \qquad (2)$$

π_i: a ranking of the elements of S_i,
$\pi_0|_{S_i}$: the ranking of the elements of S_i according to π_0.

Results

- (Pair appearance) **frequency parameter** $p \in [0, 1]$:

$$|\{S_i : S_i \ni a, b\}| \geq p \cdot |\mathcal{S}|, \forall a, b \in [n]$$

Results: Sample Complexity

- For error prob. $\epsilon \in (0,1)$: $\text{poly}(1/\beta) \cdot \Theta(\frac{1}{p} \cdot \log(n/\epsilon))$.
- **Upper Bound:** $O(\frac{1}{p(1-e^{-\beta})^2} \cdot \log(n/\epsilon))$

  Large enough sample profile length

  $\Downarrow$

  Each pair correctly ranked by majority.
- **Lower Bound:** $\Omega(\frac{1}{\beta p} \cdot \log(n/\epsilon))$

  Small sample profile length

  $\Downarrow$

  $\exists$ rarely observed, large set of disjoint pairs.

Results: Maximum Likelihood Estimation

- As in (complete) Mallows: MLE of π_0 linked to an NP-hard problem.
- However:
  - Average case $\Rightarrow$ initial approximation of MLE.
  - Structure of problem $\Rightarrow$ efficient local search.
- Finally: For any $\alpha > 0$, we find MLE w.p. $\geq 1 - n^{-\alpha}$, in time:

$$T = O\left(n^2 + n^{1+O(\frac{2+\alpha}{r\beta p^4})} 2^{O(\frac{1}{p^3\beta})} \log^2 n\right)$$

Results: Top$-k$ Retrieval

- **Upper Bound:** $O\left(\frac{\log(k/\epsilon)}{p(1-e^{-\beta})^2} + \frac{\log(n/\epsilon)}{p^2\beta k}\right)$ (if $k = \omega(1/(p\beta))$)

  Low ranked alternatives ruled out with few samples.
- **Lower Bound:** $\Omega(\frac{1}{\beta p}\log(k/\epsilon))$

  Implied by sample complexity of finding π_0.

Positional Estimator

- Given a selective Mallows sample profile, PosEst:
  1. Calculates pairwise majority positions:

$$\hat{\pi}(a) = \{b \in [n] : b < a \text{ in most common appearances}\}$$

  2. Breaks ties uniformly.
- Optimal sample complexity for π_0, Upper bound for top$-k$ & Approximates π_0 (first step of MLE).

Experiments

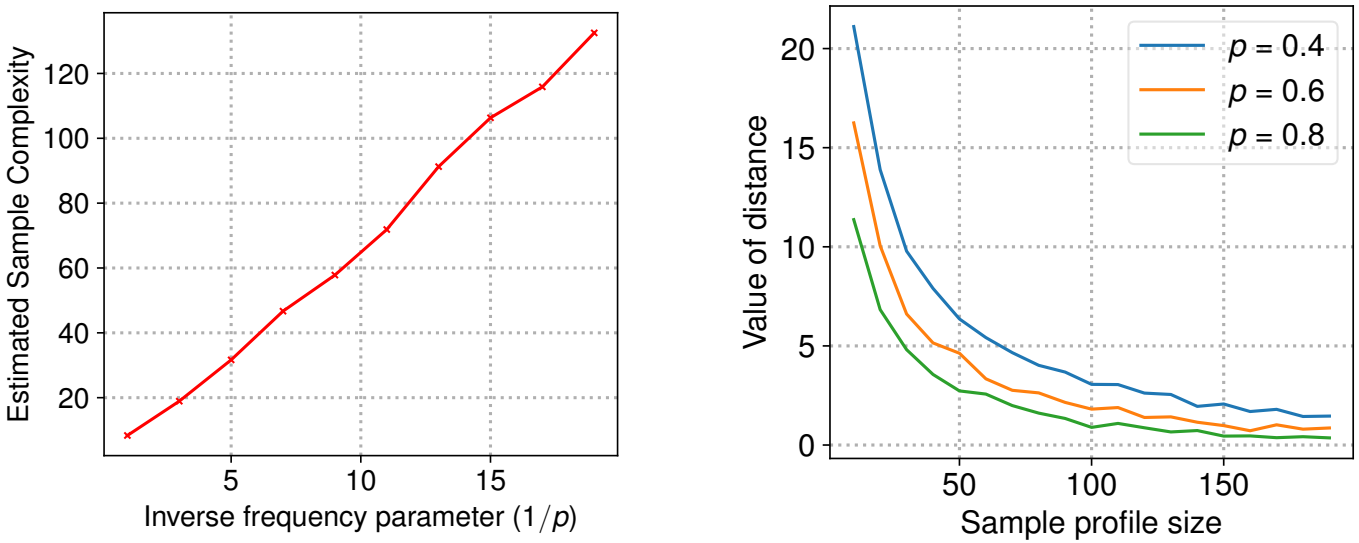


Figure: (Left:) Estimated sample complexity of retrieving, with probability at least 0.95 and using PosEst, the central ranking from selective Mallows samples, with $n = 20$, $\beta = 2$, over the frequency parameter's inverse. (Right:) Average Kendall Tau distance between the output of PosEst and the central ranking with respect to the size of the sample profile, for different values of the frequency parameter p, when $n = 20$, $\beta = 0.3$.

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